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Approximation Order of Bivariate Spline Interpolation

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In [G. Nürnberger and Th. Riessinger, *Numer. Math.* **71** (1995), 91–119], we developed an algorithm for constructing point sets at which unique Lagrange interpolation by spaces of bivariate splines of arbitrary degree and smoothness on uniform type triangulations is possible. Here, we show that similar Hermite interpolation sets yield (nearly) optimal approximation order. This is shown for differentiable splines of degree at least four defined on non-rectangular domains subdivided in uniform type triangles. Therefore, in practice we use Lagrange configurations which are "close" to these Hermite configurations. Applications to data fitting problems and numerical examples are given.

INTRODUCTION

We investigate bivariate spline spaces of the following type. Let a rectangle T and a partition of T into uniform subrectangles be given. We add to each subrectangle the same diagonal and denote the resulting partition by Δ^1. If we add to each subrectangle both diagonals, then the resulting partition is denoted by Δ^2. The space of functions in $C^r(T)$ such that the restriction of f to each subset of the partition is a bivariate polynomial of total degree q is denoted by $S_q^r(\Delta^i)$, $i=1, 2$. These spaces are called spaces of *bivariate splines of degree q and smoothness r* with respect to the partition Δ^i, $i=1, 2$. The results in this paper analogously hold for bivariate splines defined on certain non-rectangular domains (cf. Remark 6), where tensor products cannot be used.

In [12] (see also [11]), we developed jointly with Th. Riessinger a method for constructing point sets which admit unique Lagrange interpolation from $S_q^r(\Delta^i)$, $i=1, 2$. The aim of this paper is to define appropriate Hermite interpolation sets which can be considered as a limit case of the Lagrange interpolation sets and to show that the corresponding interpolating splines yield (nearly) optimal approximation order for $S_q^1(\Delta^1)$, $q \geqslant 4$.

More precisely, for each $f \in C^{q+1}(T)$, the interpolating spline $s_f \in S_q^1(\Delta^1)$ satisfies $\|D^i(f-s_f)\| \leqslant Kh^{\rho-i}$ for $i \in \{0, ..., \rho-1\}$, where $\rho=4$ if $q=4$, and

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$\rho = q+1$ if $q \geqslant 5$. Here h denotes the maximal sidelength of the subrectangles of the partition and the constant $K>0$ is independent of h. (In a future paper, we will prove similar results for $S_q^1(\Delta^2)$, $q \geqslant 2$.)

Our method is different from these known in the literature and works for splines of arbitrary degree $q \geqslant 4$. By using Bernstein–Bézier techniques, (nearly) optimal approximation order of interpolation was proved for the following spline spaces: Sha [15], Chui & He [2] and Zedek [17] (see also Jeeawock-Zedek & Sablonnière [8]) for $S_2^1(\Delta^2)$ and Sha [16] for $S_3^1(\Delta^1)$. Moreover, approximation order two for interpolation by $S_3^1(\Delta^2)$ was proved by Jeeawock-Zedek [7].

The type of interpolation sets used by Sha [16] is different from our configurations. The difference is that the interpolating splines in $S_q^1(\Delta^1)$ corresponding to our configurations can be computed locally by passing from one triangle to the next. For computing these splines, only small systems have to be solved instead of one large system. In practice we use Lagrange configurations which are "close" to our Hermite configurations. At the end of the paper, we give numerical examples (using up to 1700 interpolation points) including data fitting.

MAIN RESULTS

We consider bivariate spline spaces of the following type. First, the space of *bivariate polynomials of total degree q* is denoted by $\tilde{\Pi} = \text{span}\{x^i y^j: i \geqslant 0, j \geqslant 0, i+j \leqslant q\}$. (The corresponding univariate polynomial space is denoted by Π_q.) Let a rectangle $T = [a, b] \times [c, d]$ and points $a = x_0 < x_1 < \cdots < x_{n_1} = b$, $c = y_0 < y_1 < \cdots < y_{n_2} = d$ such that $x_i - x_{i=1} = h_1$, $i = 1, ..., n_1$; $y_j - y_{j-1} = h_2$, $j = 1, ..., n_2$, be given. By defining $R_{i,j} = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$, $i = 1, ..., n_1$; $j = 1, ..., n_2$, we obtain a partition of T into subrectangles $R_{i,j}$. If the diagonal from (x_{i-1}, y_{j-1}) to (x_i, y_j) is added to each subrectangle $R_{i,j}$, then we denote the resulting partition by Δ^1.

The spline spaces are defined as follows. Let integers r and q with $0 \leqslant r \leqslant q$ be given. The space $S_q^r = S_q^r(\Delta^1)$ of all functions $f \in C^r(T)$ such that the restriction to each subset of the partition Δ^1 is in $\tilde{\Pi}_q$ is called space of *bivariate splines of degree q and smoothness r*.

We now investigate interpolation by S_q^r. In contrast to the univariate case, it is a non-trivial problem to construct any set at which interpolation by S_q^r is possible. Therefore, we formulate the following problem: Determine a set $\{z_1, ..., z_N\}$ in T, where $N = \dim S_q^r$, such that for each function $f \in C(T)$, the *Lagrange interpolation problem* $s(z_i) = f(z_i)$, $i = 1, ..., N$ has a unique solution $s \in S_q^r$. Such a set $\{z_1, ..., z_N\}$ is called *Lagrange interpolation set* for S_q^r.

If we consider not only the function values of f but also partial derivatives of f, then we speak of a *Hermite interpolation problem* for the space S^r_q, and the corresponding sets are called *Hermite interpolation sets* for S^r_q.

For describing Herniate interpolation conditions, we denote by f_x and f_y the partial derivative of f for x and y, respectively. The higher partial derivatives are denoted by $f_{x^\alpha y^\beta}$. Given a point $z = (x, y) \in T$, we set

$$D^i f(z) = (f_{x^i}(z), f_{x^{i-1}y}(z), ..., f_{xy^{i-1}}(z), f_{y^i}(z)).$$

The uniform norm of f is defined by $\| f \| = \max_{z \in T} |f(z)|$ and for the derivatives, we set

$$\|D^i f\| = \max\{ \| f_{x^\alpha y^\beta} \| = \alpha \geqslant 0, \beta \geqslant 0, \alpha + \beta = i \}.$$

In the following, we construct Hermite interpolation sets for $S^1_q(\Delta^1)$, $q \geqslant 4$. This is done by describing Lagrange interpolation sets for these spaces and then "taking limits." The following construction of Lagrange interpolation sets is a special case of the algorithms of Nürnberger & Riessinger [12].

Construction of Lagrange Interpolation Sets

For constructing Lagrange interpolation sets for $S^1_q(\Delta^1)$, $q \geqslant 4$, we only have to describe four basic steps. For an arbitrary subtriangle V of the partition Δ^1, one of the following four steps will be applied to V.

Step A. (Starting step) Choose $q+1$ disjoint line segments $a_1, ..., a_{q+1}$ in V. For $i = 1, ..., q+1$, choose $q+2-i$ distinct points on a_i.

Step B. Choose $q-1$ disjoint line segments $b_1, ..., b_{q-1}$ in V. For $i = 1, ..., q-1$, choose $q-i$ distinct points on b_i.

Step C. Choose $q-2$ disjoint line segments $c_1, ..., c_{q-2}$ in V. For $i = 1, ..., q-2$, choose $q-i$ distinct points on c_i.

Step D. Choose $q-3$ disjoint line segments $d_1, ..., d_{q-3}$ in V. For $i = 1, ..., q-3$, choose $q-i-2$ distinct points on d_i.

Given a partition Δ^1, the construction of interpolation sets by applying the above steps successively to the subtriangles is as follows. We choose diagonal (respectively horizontal) line segments in the upper (respectively lower) triangle of each subrectangle as follows; except in the first triangle of the upper row, where we choose horizontal line segments (see Fig. 2). The points chosen on these line segments shall not lie on the triangles already considered.

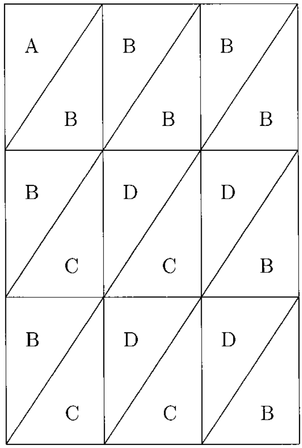

FIG. 1. Interpolation conditions for $S_q^1(\Delta^1)$.

First, we apply Step A to the first triangle (starting triangle) of the upper row of the partition Δ^1. Then passing from left to right, we apply Step B to other triangles of the upper row.

Then we consider the next row. We apply Step B to the first and the last triangle of this row, and passing from left to right, we alternatingly apply Step C and D to the remaining triangles in this row.

Then we consider the next row and apply the same steps as in the row before. We continue this method until all rows of the partition are considered (see Fig. 1).

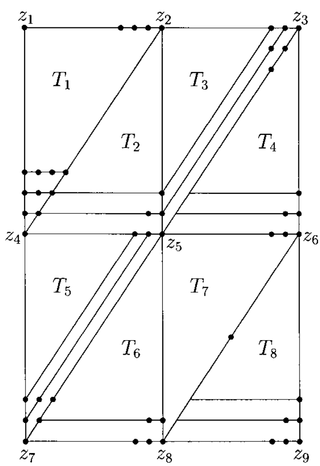

FIG. 2. Interpolation set for $S_4^1(\Delta^1)$.

(Note, that the order of the steps in the starting row (upper row) is different from the steps in all other rows.)

Next, we construct Hermite interpolation sets for $S_q^1(\Delta^1)$, $q \geqslant 4$. This is done by using the Lagrange interpolation sets above and by "taking limits." We consider the Lagrange configurations and let certain points and line segments coincide. (Fig. 2 indicates which points and line segments shall coincide.) Roughly speaking, the corresponding new interpolation conditions are obtained as follows. If certain points on some line segment coincide, then we pass to the directional derivatives along the line segment, and if certain line segments coincide, then we pass to the directional derivatives orthogonal to the line segment. In this way, we obtain the following Hermite interpolation problem.

Construction of Hermite Interpolation Sets

Let a sufficiently differentiable function $f \in C(T)$ be given. For defining Hermite interpolation conditions for a spline $s_q^1(\Delta^1)$, $q \geqslant 4$, we only have to describe four basic conditions. Let V be an arbitrary subtriangle of the partition Δ^1 and denote by U the adjacent subtriangle left of V in the same row (if it exists). One of the four following conditions will be imposed on the polynomial $p = s|_V \in \tilde{\Pi}_q$.

Condition A. (Starting condition) $p(z_1) = f(z_1)$, $p_{x^i}(z_2) = f_{x^i}(z_2)$, $i = 0, ..., q-1$, $D^i p(z_4) = D^i f(z_4)$, $i = 0, ..., q-1$, where z_1, z_2, z_4 are the vertices of the first triangle in the upper row (see Fig. 2).

Condition B. $D^i p(z) = D^i f(z)$, $i = 0, ..., q-2$, where z is the vertex of V not belonging to U.

Condition C. $D^i p(z) = D^i f(z)$, $i = 0, ..., q-2$, except $p_{y^{q-2}}(z) = f_{y^{q-2}}(z)$ where z is the vertex of V not belonging to U.

Conditiona D. $D^i p(\bar{z}) = D^i f(\bar{z})$, $i = 0, ..., q-4$, where $\bar{z}$ is the midpoint of the diagonal of V.

Given a partition Δ^1, we impose interpolation conditions on s by passing from the upper to the lower row, and by passing from the first to the last triangle in each row as follows (see Fig. 1).

First, we assign Condition A to the first triangle in the upper row of the partition Δ^1. Then passing from left to right, we assign Condition B to the remaining triangles of the upper row.

Then we consider the next row. We assign Condition B to the lower vertex z of the first triangle in this row. Then passing from left to right, we alternatingly assign Condition C and Condition D to the remaining triangles in the row, except that to the last triangle we assign Condition B.

Then we consider the next row and assign the same conditions as in the row before. We continue this method until all rows of the partition are considered.

(Note, that the order of the conditions in the starting row (upper row) is different from the conditions in all other rows.)

In the following, we will show that the spline satisfying these Hermite interpolation conditions is uniquely determined (Theorem 4) and yields (nearly) optimal approximation order (Theorem 5).

The difficulties in proving these results come from the fact that—in contrast to the finite element method (see e.g. Ciarlet [4], Ciarlet and Raviart [5])—the polynomial pieces of the interpolating spline do not satisfy $\dim \tilde{\Pi}_q$ interpolation conditions (except for the starting triangle). For example, in the case of $S^1_4(\Delta^1)$, to most of the triangles only one respectively five interpolation conditions are assigned (see Figs. 1 and 2), while $\dim \tilde{\Pi}_4 = 15$.

Therefore, one of the main principles in the proof of Theorem 5 is to show that the interpolating spline satisfies $\dim \tilde{\Pi}_q$ so-called weak interpolation conditions on each subtriangle (see Definition 3). Then Theorem 5 follows from an auxiliary result on weak interpolation by bivariate polynomials, given next.

Let a triangle W with vertices $(0, 0)$, $(\lambda_1, 0)$ and (λ_2, λ_3), where $\lambda_3 > 0$, be given. Moreover, let $0 \leqslant y_0 \leqslant \cdots \leqslant y_q \leqslant \lambda_3$ and for each $j \in \{0, ..., q\}$, $x_{0,j} \leqslant \cdots \leqslant x_{q-j,j}$ be given such that all points $z_{i,j} = (x_{i,j}, y_j)$ are contained in W. To each point $z_{i,j}$, we assign integers

$$\alpha_{i,j} = \max\{\alpha\colon x_{i-\alpha,j} = \cdots = x_{i,j}\}$$

and

$$\beta_j = \max\{\beta\colon y_{j-\beta} = \cdots = y_j\}.$$

The following result on weak interpolation holds.

LEMMA 1. *Let a function $f \in C^{q+1}(W)$, a set of bivariate polynomials $\{p_h \in \tilde{\Pi}_q\colon h \in (0, 1]\}$ and an integer σ with $1 \leqslant \sigma \leqslant q+1$ be given. If there exists a constant $K > 0$ such that for all $h \in (0, 1]$,*

$$|(f - p_h)_{x^{\alpha_{i,j}} y^{\beta_j}}(hz_{i,j})| \leqslant K h^{\sigma - \alpha_{i,j} - \beta_j}, \; i = 0, ..., q-j; j = 0, ..., q, \tag{1}$$

then there exists a constant $\tilde{K} > 0$ such that for all $h \in (0, 1]$ and $\omega \in \{0, ..., \sigma - 1\}$,

$$\|D^{\omega}(f - p_h)\|_{hW} \leqslant \tilde{K} h^{\sigma - \omega}. \tag{2}$$

(The constant $\tilde{K} > 0$ depends on K, q, $\|D^{q+1} f\|$, the smallest angle of W and is independent of h.)

Proof. It is well known (see e.g. Chui [1]) that for all $h \in (0, 1]$, there exists a unique polynomial $\tilde{p}_h \in \tilde{\Pi}_q$ which satisfies the interpolation conditions

$$(\tilde{p}_h)_{x^{\alpha_{i,j}} y^{\beta_j}}(hz_{i,j}) = f_{x^{\alpha_{i,j}} y^{\beta_j}}(hz_{i,j}),\ i = 0, ..., q-j;\ j = 0, ..., q. \tag{3}$$

It follows from Theorem 4 in Ciarlet & Raviart [5] that there exists a constant $C_1 > 0$ such that for all $h \in (0, 1]$ and $\omega \in \{0, ..., q\}$,

$$\|D^{\omega}(f - \tilde{p}_h)\|_{hW} \leqslant C_1 h^{q+1-\omega},$$

where C_1 depends on q, $\|D^{q+1} f\|$, the smallest angle of W and is independent of h. Therefore, we get

$$\begin{aligned}\|D^{\omega}(f - p_h)\|_{hW} &\leqslant \|D^{\omega}(f - \tilde{p}_h)\|_{hW} + \|D^{\omega}(\tilde{p}_h - p_h)\|_{hW} \\ &\leqslant C_1 h^{q+1-\omega} + \|D^{\omega}(\tilde{p}_h - p_h)\|_{hW}.\end{aligned}$$

We set $Q_h = \tilde{p}_h - p_h \in \tilde{\Pi}_q$ and have to show that there exists a constant $C_2 > 0$ (independent of h) such that for all $h \in (0, 1]$ and $\omega \in \{0, ..., \sigma - 1\}$,

$$\|D^{\omega} Q_h\|_{hW} \leqslant C_2 h^{\sigma - \omega}. \tag{4}$$

Since the interpolating polynomials considered here are uniquely determined, the polynomial Q_h can be written in the form

$$Q_h(z) = \sum_{\substack{i = 0, ..., q-j; \\ j = 0, ..., q}} L_{h,i,j}(z) (Q_h)_{x^{\alpha_{i,j}} y^{\beta_j}}(hz_{i,j}), \tag{5}$$

where $L_{h,i,j}$ are the fundamental polynomials satisfying the interpolation conditions

$$(L_{h,i,j})_{x^{\alpha_{\mu,v}} y^{\beta_v}}(hz_{\mu,v}) = \delta_{(i,j),(\mu,v)},$$

where $\delta_{(i,j),(\mu,v)}$ is 1 if $(i, j) = (\mu, v)$, and 0 if $(i, j) \neq (\mu, v)$, for $\mu = 0, ..., q - v$; $v = 0, ..., q$. Moreover, for all $z \in hW$,

$$L_{h,i,j}(z) = h^{\alpha_{i,j} + \beta_j} L_{1,i,j}\left(\frac{1}{h} z\right). \tag{6}$$

This equation holds, since the polynomial on the right side of (6) satisfies the same interpolation conditions as $L_{h,i,j}$. It follows from assumption (1) that

$$\begin{aligned}(Q_h)_{x^{\alpha_{i,j}} y^{\beta_j}}(hz_{i,j})| &= |(\tilde{p}_h - p_h)_{x^{\alpha_{i,j}} y^{\beta_j}}(hz_{i,j})| \\ &= |(f - p_h)_{x^{\alpha_{i,j}} y^{\beta_j}}(hz_{i,j})| \\ &\leqslant K h^{\sigma - \alpha_{i,j} - \beta_j}.\end{aligned}$$

Then it follows from (5) and (6) that for all $\omega \in \{0, ..., \sigma-1\}$,

$$\begin{aligned}\|D^{\omega} Q_h\|_{hW} &\leqslant \sum_{\substack{i=0, ..., q-j;\\ j=0, ..., q}} Kh^{\sigma-\alpha_{i,j}-\beta_j} \|D^{\omega} L_{h,i,j}\|_{hW} \\ &= \sum_{\substack{i=0, ..., q-j;\\ j=0, ..., q}} Kh^{\sigma-\alpha_{i,j}-\beta_j} h^{\alpha_{i,j}+\beta_j-\omega} \|D^{\omega} L_{1,i,j}\|_{W} \\ &= \left(K \sum_{\substack{i=0, ..., q-j;\\ j=0, ..., q}} \|D^{\omega} L_{1,i,j}\|_{W}\right) h^{\sigma-\omega}.\end{aligned}$$

By denoting the term in brackets by C_2, we get (4). This proves Lemma 1. ∎

Remark 2. (i) The proof of Lemma 1 shows that Lemma 1 also holds if $(0, 1]$ is replaced by an arbitrary subset of $(0, 1]$.

(ii) Moreover, a univariate version of Lemma 1 holds for $f \in C^{q+1}[0, 1]$ (with the same proof): Let a set of univariate polynomials $\{g_h \in \Pi_q : h \in (0, 1]\}$, points $0 \leqslant t_0 \leqslant \cdots \leqslant t_q \leqslant 1$ and an integer σ with $1 \leqslant \sigma \leqslant q+1$ be given. For $\mu \in \{0, ..., q\}$, we set $\gamma_\mu = \max\{\gamma : t_{\mu-\gamma} = \cdots = t_\mu\}$. If there exists a constant $C > 0$ such that for all $h \in (0, 1]$,

$$|(f - g_h)^{(\gamma_\mu)}(ht_\mu)| \leqslant Ch^{\sigma-\gamma_\mu}, \mu = 0, ..., q,$$

then there exists a constant $\tilde{C} > 0$ such that for all $h \in (0, 1]$ and $\omega \in \{0, ..., \sigma-1\}$,

$$\|(f - g_h)^{(\omega)}\|_{[0, h]} \leqslant \tilde{C} h^{\sigma-\omega}.$$

For simplicity, we use the following definition.

Definition 3. We say that a set of bivariate polynomials $\{p_h \in \tilde{\Pi}_q : h \in (0, 1])\}$ *weakly interpolates* f on W if there exists a set of points $\{z_{i,j} : i = 0, ..., q-j; j = 0, ..., q\}$ as in Lemma 1 such that (1) holds with $\sigma = q+1$. If the context is clear, then we simply say that $p_h \in \tilde{\Pi}_q$ *weakly interpolates* f on hW. Moreover, in this case we also say that $(p_h)_{y^{\beta_j}} \in \tilde{\Pi}_{q-\beta_j}$ *weakly interpolates* $f_{y^{\beta_j}}$ on the line segment $\{(x, y) : y = hy_j\} \cap hW$, $j = 0, ..., q$.

We now show that the spline satisfying the Hermite interpolation conditions above (see Conditions A-C) is uniquely determined.

Theorem 4. *For each sufficiently differentiable function $f \in C(T)$, there exists a unique spline $s_f \in S^1_q(\Delta^1)$, $q \geqslant 4$, which satisfies the Hermite interpolation conditions above.*

Proof. Let a spline $s \in S_q^1(\Delta^1)$, $q \geqslant 4$, be given which satisfies the homogeneous interpolation conditions. By applying the arguments in the proof of Theorem 5, we can show that $s = 0$ on T. In the proof of Theorem 5, it will be shown that the interpolating spline satisfies dim $\tilde{\Pi}_q$ weak interpolation conditions on each subtriangle of Δ^1. By using the same arguments, it follows in the case of homogeneous interpolation conditions that s satisfies dim $\tilde{\Pi}_q$ homogeneous interpolation conditions on each subtriangle which implies that $s = 0$ on the subtriangles. This is done as follows. First, it follows that $s = 0$ on the first triangle of the upper row of the partition Δ^1. Then passing from left to right, it follows that $s = 0$ on the remaining triangles of the upper row. Then we consider the next row. It follows that $s = 0$ on the first triangle of this row. Again passing from left to right, it follows that $s = 0$ on the remaining triangles of this row. By proceeding in this way, we get that $s = 0$ on T. This proves Theorem 4. ∎

The next result shows that our Hermite interpolation method yields (nearly) optimal approximation order. We denote by γ the angle between the horizontal and diagonal lines of the partition Δ^1. Moreover, we set $h = \max\{h_1, h_2\}$, where $h_1 = x_i - x_{i-1}$, $i = 1, ..., n_1$, and $h_2 = y_j - y_{j-1}$, $j = 1, ..., n_2$. In Theorem 5, the norm denotes the maximum of the uniform norm over all subtriangles of the partition (w.r.t. the polynomial pieces).

THEOREM 5. *For each function $f \in C^{q+1}(T)$, there exists a constant $K > 0$ such that for the unique interpolating spline $s_f \in S_q^1(\Delta^1)$ in Theorem 4 and for all $i \in \{0, ..., \rho - 1\}$,*

$$\|D^i(f - s_f)\| \leqslant K h^{\rho - i},$$

where $\rho = 4$ if $q = 4$, and $\rho = q + 1$ if $q \geqslant 5$. (The constant $K > 0$ depends on q, γ, $\|D^{q+1} f\|$ and is independent of h.)

Proof. Let a partition Δ^1 of T be given. The partition Δ^1 depends on h. The proof will show that it suffices to consider the partition of Fig. 2. Let $s_f \in S_q^1(\Delta^1)$, $q \geqslant 4$, be the unique interpolating spline of f. The spline $s_{f,h} = s_f$ and each subtriangle $T_{i,h} = T_i$ of the partition depends on h. We first consider the case when $q \geqslant 5$. We consider each subtriangle $T_{i,h}$ separately and may assume that it is of the form as in Lemma 1. The method of proof is to show that for each subtriangle $T_{i,h}$, the polynomial $p_{i,h} = s_{f,h}|_{T_{i,h}} \in \tilde{\Pi}_q$ weakly interpolates f on $T_{i,h}$. Since only special values of h can occur, we apply Lemma 1 in the sense of Remark 2, (i). Then it follows that Theorem 5 holds for $q \geqslant 5$. For simplicity we write T_i, s_f and p_i instead of $T_{i,h}$, $s_{f,h}$ and $p_{i,h}$. Thus we have to show

Claim. *For each subtriangle T_i, the polynomial $p_i = s_f|_{T_i} \in \tilde{\Pi}_q$ weakly interpolates f on T_i.*

We start with the subtriangle T_1. The Claim is true for T_1, since $p_1 \in \tilde{\Pi}_q$ even interpolates f on T_1. Next, we consider the subtriangle T_2. In the following, we will use the fact that certain higher derivatives (in direction of r) of p_1 and p_2 coincide, although s_f is only in $C_1(T)$. We denote by $r = (r_1, r_2)$ the unit vector in direction of the diagonal and by $r^\perp = (-r_2, r_1)$. First, we show

Claim 1. *For all $\alpha \in \{0, 1\}$, $(p_2)_{(r^\perp)^\alpha} \in \tilde{\Pi}_{q-\alpha}$ weakly interpolates $f_{(r^\perp)^\alpha}$ on $[z_2, z_4]$.*

Proof. We first note that for all $\alpha \in \{0, ..., q\}$, $(p_1)_{r^\alpha} = (p_2)_{r^\alpha}$ and $(p_1)_{r^\perp r^\alpha} = (p_2)_{r^\perp r^\alpha}$ on the diagonal between T_1 and T_2, since $s_f \in C^1(T)$. The fact that similar statements hold for all pairs of adjacent triangles is used in the arguments below. First, it follows from the interpolation conditions that $p_2 \in \tilde{\Pi}_q$ interpolates f on $[z_2, z_4]$. Then it follows from the univariate version of Lemma 1 (see Remark 2) that for all $\alpha \in \{0, ..., q\}$,

$$\|(f - p_2)_{r^\alpha}\|_{[z_2, z_4]} \leqslant K_1 h^{q+1-\alpha}$$

for some constant $K_1 > 0$. Therefore,

$$|(f - p_2)_{r^\perp}(z_2)| = \left| -\frac{1}{r_2}(f - p_2)_x(z_2) + \frac{r_1}{r_2}(f - p_2)_r(z_2) \right| \leqslant \frac{r_1}{r_2} K_1 h^q.$$

(Here and in the following, we use that for $F \in C^\lambda(T)$,

$$F_{(\alpha_1 R_1 + \alpha_2 R_2)^\lambda} = \sum_{\mu=0}^{\lambda} \binom{\lambda}{\mu} \alpha_1^{\lambda-\mu} \alpha_2^\mu F_{R_1^{\lambda-\mu} R_2^\mu},$$

where R_1, R_2 and $\alpha_1 R_1 + \alpha_2 R_2$ are unit vectors and λ is a natural number.) Then by the interpolation conditions of p_2 at z_4 we get that $(p_2)_{r^\perp} \in \tilde{\Pi}_{q-1}$ weakly interpolates $f_{r^\perp}$ on $[z_2, z_4]$. This proves Claim 1.

By using Claim 1, we will show

Claim 2. *For all $\alpha \in \{0, ..., q-2\}$, $(p_2)_{y^\alpha} \in \tilde{\Pi}_{q-\alpha}$ weakly interpolates f_{y^α} on $[z_4, z_5]$.*

Proof. We prove Claim 2 by induction on α. First, it follows from the interpolation conditions that Claim 2 holds for $\alpha = 0$. We assume that Claim 2 holds for $\alpha \in \{0, ..., j\}$, $j \leqslant q-3$, and show that Claim 2 holds for

$j+1$. For doing this, we will show that for all α and β with $\alpha+\beta=j+1$ and $\alpha+\beta=j+2$,

$$|(f-p_2)_{r^\alpha x^\beta}(z_4)| \leqslant K_2 h^{q+1-\alpha-\beta} \tag{7}$$

for some constant $K_2>0$. First, we assume that (7) holds. Then it follows that

$$\begin{aligned} |(f-p_2)_{y^{j+1}}(z_4)| &= \left| \sum_{\nu=0}^{j+1} (-1)^\nu \binom{j+1}{\nu} \left(\frac{1}{r_2}\right)^{j+1-\nu} \left(\frac{r_1}{r_2}\right)^\nu (f-p_2)_{r^{j+1-\nu}x^\nu}(z_4) \right| \\ &\leqslant 2^{j+1} \left(\frac{1}{r_2}\right)^{j+1} K_2 h^{q-j}. \end{aligned}$$

Moreover, we get

$$\begin{aligned} &|(f-p_2)_{y^{j+1}x}(z_4)| \\ &\quad = \left| \sum_{\nu=0}^{j+1} (-1)^\nu \binom{j+1}{\mu} \left(\frac{1}{r_2}\right)^{j+1-\nu} \left(\frac{r_1}{r_2}\right)^\nu (f-p_2)_{r^{j+1-\nu}x^{\nu+1}}(z_4) \right| \\ &\quad \leqslant 2^{j+1} \left(\frac{1}{r_2}\right)^{j+1} K_2 h^{q-j-1}. \end{aligned}$$

It follows from these inequalities and the interpolation conditions that Claim 2 holds for $\alpha=j+1$.

Therefore, it remains to show (7). First, it follows from Claim 1 and Lemma 1 (univariate version) that for all $\mu \in \{0, 1\}$ and $\nu \in \{0, ..., q-1\}$,

$$|(f-p_2)_{(r^\perp)^\mu r^\nu}(z_4)| \leqslant K_3 h^{q+1-\mu-\nu}$$

for some constant $K_3>0$. Then it follows that for all $\alpha \in \{j, j+1\}$,

$$|(f-p_2)_{r^\alpha x}(z_4)| = |r_1(f-p_2)_{r^{\alpha+1}}(z_4) - r_2(f-p_2)_{r^\alpha r^\perp}(z_4)| \leqslant 2K_3 h^{q-\alpha}.$$

Now, let $\beta \geqslant 2$ and $\alpha \leqslant j$ be given. Then it follows from the induction hypothesis and Lemma 1 (univariate version) that for all $\mu \leqslant j$ and $\nu \leqslant q-j$

$$\|(f-p_2)_{y^\mu x^\nu}\|_{[z_4, z_5]} \leqslant K_4 h^{q+1-\mu-\nu}$$

for some constant $K_4>0$. This implies that

$$|(f-p_2)_{r^\alpha x^\beta}(z_4)| = \left| \sum_{\mu=0}^{\alpha} \binom{\alpha}{\mu} r_2^\mu r_1^{\alpha-\mu} (f-p_2)_{y^\mu x^{\alpha-\mu+\beta}}(z_4) \right| \leqslant 2^\alpha K_4 h^{q+1-\alpha-\beta}.$$

This proves Claim 2.

By using Claim 1 and Lemma 1 (univariate version), we get

$$|(f-p_2)_y(z_2)| = \left| -\frac{r_1}{r_2}(f-p_2)_x(z_2) + \frac{1}{r_2}(f-p_2)_r(z_2) \right| \leqslant K_5 h^q \tag{8}$$

for some constant $K_5 > 0$. Since by the interpolation conditions $(f-p_2)(z_2)=0$ and $(f-p_2)_x(z_2)=0$, it follows from (8) and Claim 2 that the Claim is true for T_2. Next, we consider the subtriangle T_3 and argue analogously as for T_2. We first prove

CLAIM 3. *For all* $\alpha \in \{0, 1\}$, $(p_3)_{x^\alpha} \in \tilde{\Pi}_{q-\alpha}$ *weakly interpolates* f_{x^α} *on* $[z_2, z_5]$.

Proof. First, it follows from (8) and the interpolation conditions at z_5 that the claim is true for $\alpha = 0$. Moreover, by Claim 1 and Lemma 1 (univariate version)

$$|(f-p_2)_{ry}(z_2)| = |r_2(f-p_2)_{rr}(z_2) + r_1(f-p_2)_{rr^\perp}(z_2)| \leqslant K_6 h^{q-1}$$

for some constant $K_6 > 0$. Since $p_2 \in \tilde{\Pi}_q$ weakly interpolates f on $[z_2, z_5]$, it follows from Lemma 1 (univariate version) that

$$|(f-p_2)_{yx}(z_2)| = \left| \frac{1}{r_1}(f-p_2)_{yr}(z_2) - \frac{r_2}{r_1}(f-p_2)_{yy}(z_2) \right| \leqslant K_7 h^{q-1}$$

for some constant $K_7 > 0$. Therefore, it follows from the interpolation conditions at z_2 and z_5 that Claim 3 is true for $\alpha = 1$.

By using Claim 3, we can show analogously as in the proof of Claim 2 that for all $\alpha \in \{0, ..., q-2\}$ and $\beta \in \{0, 1\}$,

$$|(f-p_3)_{(r^\perp)^\alpha r^\beta}(z_5)| \leqslant K_8 h^{q+1-\alpha-\beta}$$

for some constant $K_8 > 0$. Together with the interpolation conditions at z_3, we get

CLAIM 4. *For all* $\alpha \in \{0, ..., q-2\}$, $(p_3)_{(r^\perp)^\alpha} \in \tilde{\Pi}_{q-\alpha}$ *weakly interpolates* $f_{(r^\perp)^\alpha}$ *on* $[z_3, z_5]$.

By using (8) and the interpolation conditions at z_2, we get

$$(f-p_3)(z_2) = 0, \ |(f-p_3)_r(z_2)| \leqslant K_9 h^q \quad \text{and} \quad |(f-p_3)_{r^\perp}(z_2)| \leqslant K_9 h^q$$

for some constant $K_9 > 0$. This shows that the Claim is true for T_3. Next, we consider the subtriangle T_4. Analogously as above by using Claim 4 (for $\alpha = 0, 1$), we get

CLAIM 5. *For all* $\alpha \in \{0, ..., q-2\}$, $(p_4)_{y^\alpha} \in \tilde{\Pi}_{q-\alpha}$ *weakly interpolates* f_{y^α} *on* $[z_5, z_6]$. (*In particular, for all* $\alpha \in \{0, ..., q-2\}$ *and* $\beta \in \{0, 1\}$, $|(f-p_4)_{y^\alpha x^\beta}(z_5)| \leqslant K_{10} h^{q+1-\alpha-\beta}$ *for some constant* $K_{10} > 0$.)

This together with the interpolation conditions at z_3 shows that the Claim is true for T_4.

Next, we consider the subtriangle T_5. Analogously as above, by using Claim 2 (for $\alpha = 0, 1$), we get

CLAIM 6. *For all* $\alpha \in \{0, ..., q-2\}$, $(p_5)_{(r^\perp)^\alpha} \in \tilde{\Pi}_{q-\alpha}$ *weakly interpolates* $f_{(r^\perp)^\alpha}$ *on* $[z_5, z_7]$. (*In particular, for all* $\alpha \in \{0, ..., q-2\}$ *and* $\beta \in \{0, 1\}$, $|(f-p_5)_{(r^\perp)^\alpha r^\beta}(z_5)| \leqslant K_{11} h^{q+1-\alpha+\beta}$ *for some constant* $K_{11} > 0$.)

This together with the interpolation conditions at z_4 shows that the Claim is true for T_5.

Next, we consider the subtriangle T_6. The Claim for T_6 can be shown analogously as for T_4 with the only exception that for T_4 we have $(f-p_4)_{y^{q-2}}(z_6) = 0$, while for T_6 the condition $(f-p_6)_{y^{q-2}}(z_8) = 0$ is not given. On the other hand, it suffices to show that

$$|(f-p_6)_{y^{q-2}}(z_8)| \leqslant K_{12} h^3 \tag{9}$$

for some constant $K_{12} > 0$. This is done as follows. We first show

CLAIM 7.

$$|(f-p_6)_{yy}(z_5)| \leqslant |(f-p_2)_{yy}(z_5)| + K_{13} h^{q-1}$$

for some constant $K_{13} > 0$.

Proof.

$$\begin{aligned}
&(f-p_6)_{yy}(z_5) \\
&\quad = \frac{1}{r_2}(f-p_6)_{yr}(z_5) - \frac{r_1}{r_2}(f-p_7)_{yx}(z_5) \\
&\quad = \frac{1}{r_2}\left(\frac{1}{r_2}(f-p_5)_{rr}(z_5) - \frac{r_1}{r_2}(f-p_5)_{xr}(z_5)\right) \\
&\qquad - \frac{r_1}{r_2}\left(-\frac{r_1}{r_2}(f-p_4)_{xx}(z_5) - \frac{1}{r_2}(f-p_4)_{rx}(z_5)\right) \\
&\quad = \frac{1}{r_2}\left(\frac{1}{r_2}(f-p_5)_{rr}(z_5) - \frac{r_1}{r_2}(r_1(f-p_2)_{xx}(z_5) + r_2(f-p_2)_{xy}(z_5))\right) \\
&\qquad - \frac{r_1}{r_2}\left(-\frac{r_1}{r_2}(f-p_4)_{xx}(z_5) - \frac{1}{r_2}\left(\frac{1}{r_1}(f-p_3)_{rr}(z_5) - \frac{r_2}{r_1}(f-p_3)_{ry}(z_5)\right)\right)
\end{aligned}$$

$$=\frac{1}{r_2}\left(\frac{1}{r_2}(f-p_5)_{rr}(z_5)-\frac{r_1}{r_2}(r_1(f-p_2)_{xx}(z_5)+r_2(f-p_2)_{xy}(z_5))\right)$$

$$-\frac{r_1}{r_2}\left(-\frac{r_1}{r_2}(f-p_4)_{xx}(z_5)-\frac{1}{r_2}\left(\frac{1}{r_1}(f-p_3)_{rr}(z_5)\right.\right.$$

$$\left.\left.-\frac{r_2}{r_1}[r_2(f-p_2)_{yy}(z_5)+r_1(f-p_2)_{xy}(z_5)]\right)\right)$$

Since s_f satisfies $q+1$ interpolation conditions on each edge of the partition which contains z_5 (except on $[z_5, z_8]$), it follows from Lemma 1 (univariate version) that the above second partial derivatives are bounded by h^{q-1} up to some constant. Since $(f-p_2)_{xy}(z_5)=0$, it follows that there exists a constant $K_{13}>0$ such that

$$|(f-p_6)_{yy}(z_5)| \leqslant |(f-p_2)_{yy}(z_5)| + K_{13}h^{q-1}.$$

This proves Claim 7.

Since $(f-p_2)_{yy}(z_5)=0$, it follows from Claim 7 that $|(f-p_6)_{yy}(z_5)| \leqslant K_{13}h^{q-1}$. This together with the interpolation conditions at z_5 and z_8 shows that $p_6 \in \tilde{\Pi}_q$ weakly interpolates f on $[z_5, z_8]$. Therefore, it follows from Lemma 1 (univariate version) that (9) holds. This proves the Claim for T_6.

Next, we consider the subtriangle T_7. From Claim 5 we get

CLAIM 8. *For all* $\alpha \in \{0, 1\}$, $(p_7)_{y^\alpha} \in \tilde{\Pi}_{q-\alpha}$ *weakly interpolates* f_{y^α} *on* $[z_5, z_6]$. *Moreover, from the proof of* (9) *and the interpolation conditions at* z_5 *and* z_6 *follows*

CLAIM 9. *For all* $\alpha \in \{0, 1\}$, $(p_7)_{x^\alpha} \in \tilde{\Pi}_{q-\alpha}$ *weakly interpolates* f_{x^α} *on* $[z_5, z_8]$. *By using Claims* 8 *and* 9, *analogously as above we can show*

CLAIM 10. *For all* $\alpha \in \{0, ..., q-2\}$, $(p_7)_{(r^\perp)^\alpha} \in \tilde{\Pi}_{q-\alpha}$ *weakly interpolates* $f_{(r^\perp)^\alpha}$ *on* $[z_6, z_8]$. (*In particular, for all* $\alpha \in \{0, ..., q-2\}$ *and* $\beta \in \{0, 1\}$, $|(f-p_7)_{(r^\perp)^\alpha r^\beta}(z_6)| \leqslant K_{14}h^{q+1-\alpha-\beta}$ *and* $|(f-p_7)_{(r^\perp)^\alpha r^\beta}(z_8)| \leqslant K_{14}h^{q+1-\alpha-\beta}$ *for some constant* $K_{14}>0$.) *This together with the interpolation conditions at* z_5 *shows that the Claim is true for* T_7. *Finally, the Claim for* T_8 *follows analogously as for* T_4.

Now, for a general partition we argue as follows. We first consider the upper row. By passing from left to right, we apply the arguments for $T_1, ..., T_4$. Then we consider the next row. We apply the arguments for T_5 to the first triangle of this row. Then we alternatingly apply the arguments for T_6 and T_7 to the remaining triangles of this row except to the last triangle. We apply the arguments for T_8 to the last triangle. Then we consider the next row and argue as in the row before until all rows are considered. This proves Theorem 5 for $q \geqslant 5$.

Finally we consider the case $q=4$. The proof for $q=4$ is completely analogous to the case $q \geqslant 5$ with the following exception. Let w_i be an interior grid point and g_j be a suitable polynomial piece of s_f such that $(g_j)_{y^{q-2}}(w_i)$ is defined. Then as shown for $q \geqslant 5$, the value $|(f-g_j)_{y^{q-2}}(w_i)|$ is bounded by h^3 up to some constant, while for $q=4$ this value is only bounded by h^2 up to some constant. This will be proved in the following. For simplicity, we consider the first column of the partition and use a new notation as indicated in Fig. 3. We set

$$h_i = s_f\,|_{V_i},\ i=1, ..., n_2.$$

It follows from the proof of Claim 7 that

$$|(f-h_{i+1})_{yy}(w_i)| \leqslant (f-h_i)_{yy}(w_i)| + K_{15}h^3,\ i=1, ..., n_2-1 \tag{10}$$

for some constant $K_{15}>0$. We will show that

$$|(f-h_{i+1})_{yy}(w_{i+1})| \leqslant |(f-h_{i+1})_{yy}(w_i)| + K_{16}h^3,\ i=1, ..., n_2-1 \tag{11}$$

for some constant $K_{16}>0$. We first assume that (11) holds. Then it follows from (10) and (11) that for all $i \in \{2, ..., n_2\}$,

$$\begin{aligned}
|(f-h_i)_{yy}(w_i)| &\leqslant |(f-h_i)_{yy}(w_{i-1})| + K_{16}h^3 \\
&\leqslant |(f-h_{i-1})_{yy}(w_{i-1})| + (K_{15}+K_{16})h^3 \\
&\leqslant \cdots \\
&\leqslant |(f-h_1)_{yy}(w_1)| + (i-1)(K_{15}+K_{16})h^3 \\
&\leqslant n_2(K_{15}+K_{16})h^3 \\
&= \frac{d-c}{h_2}(K_{15}+K_{16})h^3 = K_{17}h^2 \qquad (12)
\end{aligned}$$

for some constant $K_{17}>0$.

We finally prove (11). Let $i \in \{1, ..., n_2-1\}$ be given. Let $\tilde{h}_{i+1}$ be a polynomial in $\tilde{\Pi}_4$ such that

$$\begin{gathered}
\tilde{h}_{i+1}(w_i)=f(w_i), \quad (\tilde{h}_{i+1})_y(w_i)=f_y(w_i), \quad (\tilde{h}_{i+1})_{yy}(w_i)=f_{yy}(w_i), \\
\tilde{h}_{i+1}(w_{i+1})=f(w_{i+1}) \qquad \text{and} \qquad (\tilde{h}_{i+1})_y(w_{i+1})=f_y(w_{i+1}).
\end{gathered}$$

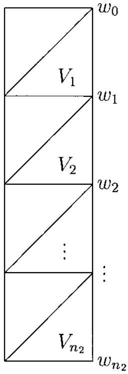

FIG. 3. First column of the partition.

We note that the polynomial is uniquely determined on $[w_i, w_{i+1}]$. It follows from the interpolation conditions for h_i that

$$(\tilde{h}_{i+1} - h_{i+1})(w_i) = 0, \qquad (\tilde{h}_{i+1} - h_{i+1})_y (w_i) = 0,$$
$$(\tilde{h}_{i+1} - h_{i+1})(w_{i+1}) = 0, \quad (\tilde{h}_{i+1} - h_{i+1})_y (w_{i+1}) = 0.$$

Therefore, for all $w \in [w_i, w_{i+1}]$,

$$(\tilde{h}_{i+1} - h_{i+1})(w) = \lambda (w - w_i)^2 (w - w_{i+1})^2$$

for some real number λ. Then it is easy to verify that

$$(\tilde{h}_{i+1} - h_{i+1})_{yy} (w_i) = (\tilde{h}_{i+1} - h_{i+1})_{yy} (w_{i+1}).$$

It follows that

$$\begin{aligned} &|(f - h_{i+1})_{yy} (w_{i+1}) - (f - h_{i+1})_{yy} (w_i)| \\ &\quad = |(f - \tilde{h}_{i+1})_{yy} (w_{i+1}) - (f - \tilde{h}_{i+1})_{yy} (w_i) \\ &\qquad + (\tilde{h}_{i+1} - h_{i+1})_{yy} (w_{i+1}) - (\tilde{h}_{i+1} - h_{i+1})_{yy} (w_i)| \\ &\quad = |(f - \tilde{h}_{i+1})_{yy} (w_{i+1})| \leqslant K_{16} h_2^3 \end{aligned}$$

for some constant $K_{16} > 0$. This inequality follows from Lemma 1 (univariate version) by using the interpolation properties of h_{i+1}. This implies that

$$|(f - h_{i+1})_{yy} (w_{i+1})| \leqslant |(f - h_{i+1})_{yy} (w_i)| + K_{16} h^3$$

and proves (11).

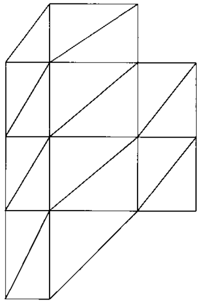

FIG. 4. Non-rectangular domain.

Finally, the claim of Theorem 5 for $q=4$ follows from (12) by applying the proof for $q \geqslant 5$ and Lemma 1. This proves Theorem 5. ∎

Remark 6. A close inspection of the proof of Theorem 5 shows the following. Theorems 4 and 5 also hold when the partitions of $T=[a, b] \times [c, d]$ are non-uniform, where $h_1 = \max\{x_i - x_{i-1}\colon i=1, ..., n_1\}$, $h_2 = \max\{y_i - y_{i-1}\colon i=1, ..., n_2\}$, $h = \max\{h_1, h_2\}$ and γ denotes the smallest angle which appears in the subtriangles of the partition. Moreover, the results also hold for splines defined on any simply connected subset of $[a, b] \times [c, d]$ which is the union of given subtriangles such that every pair of successive subtriangles has a common edge (see Fig. 4). We note that for non-rectangular domains of this type, tensor products cannot be used.

DATA FITTING

We now consider the case when only data f_i on certain points (u_i, v_i) in $T=[a, b] \times [c, d]$ are given (instead of a function $f \in C(T)$) which we want to approximate by $S_q^1(\Delta^1)$, $q \geqslant 4$. First, we describe the method for the simplest case.

We set $\tilde{q}=3$ if $q=4$, and $\tilde{q}=q$, if $q \geqslant 5$. Let points $a=u_0<u_1< \cdots < u_{m_1}=b$, $c=v_0<v_1< \cdots <v_{m_2}=d$, and a uniform partition Δ^1 of T be given such that each subtriangle of Δ^1 contains $\dim \tilde{\Pi}_{\tilde{q}} = (\tilde{q}+1)(\tilde{q}+2)/2$ points (u_i, v_i). For each point (u_i, v_i), let a real number f_i be given.

In the first step, we interpolate the given data f_i on each subtriangle by a polynomial. It is well known (see e.g. Chui [1]) that for each subtriangle T_j of Δ^1 there exists a unique $p_j \in \tilde{\Pi}_{\tilde{q}}$ such that

$$p_j(u_i, v_i) = f_i$$

for every point (u_i, v_i) in T_j. The resulting spline $\tilde{s} \in S_{\tilde{q}}^0(\Delta^1)$ is continuous, if there are $\tilde{q}+1$ interpolation points on every edge of the subtriangles.

In the second step, we interpolate the resulting function $\tilde{s}$ by a differentiable spline $s \in S_q^1(\Delta^1)$ which satisfies the Hermite interpolation conditions as in Theorem 4 for $\tilde{s}$ instead of f.

We now consider the approximation order of this method. Therefore, let a function $f \in C^{q+1}(T)$ be given and $f_i = f(u_i, v_i)$ for all points (u_i, v_i) in T. It follows from Ciarlet & Raviart [5] that there exists a constant $\tilde{K} > 0$ (independent of h) such that for all $i \in \{0, ..., \tilde{q}\}$,

$$\|D^i(f - \tilde{s})\| \leqslant \tilde{K} h^{\tilde{q}+1-i}, \tag{13}$$

where h corresponds to the partition Δ^1 (as in Theorem 5). Since s interpolates $\tilde{s}$, it follows from (13) that s interpolates f up to an error of order $\tilde{q}+1$. Now, the proof of Theorem 5 (by using Lemma 1 on weak interpolation) shows that only this is needed to get the estimate (as in Theorem 5) that for all $i \in \{0, ..., \rho - 1\}$,

$$\|D^i(f - s)\| \leqslant \tilde{K} h^{\rho - i},$$

where $\rho = 4$ if $q = 4$, and $\rho = q + 1$ if $q \geqslant 5$.

This two step method can be applied in the following more general cases. Let uniform or scattered data in T be given. If it is possible to get a piecewise polynomial $\tilde{s}$ on T which interpolates or approximates the given data up to an error of order at most $\tilde{q}+1$, then we can interpolate $\tilde{s}$ by a spline $s \in S_q^1(\Delta^1)$ and get the same approximation order for s. Moreover, this method can also be applied to simply connected subsets of T as in Remark 6.

NUMERICAL EXAMPLES

In practice, we use Lagrange configurations which are "close" to our Hermite configurations (see Fig. 2). In the computation of the interpolating spline, only small systems have to be solved instead of one large system. This is done by computing the spline on the starting triangle and then passing from one triangle to the next as in the definition of the interpolation sets.

The dimension of bivariate spline spaces of the above type was determined by Chui & Wang [3] and Schumaker [14]. For uniform partitions, a basis of such spaces was given by Chui & Wang [3] and Dahmen & Micchelli [6]. Such a basis consists of bivariate polynomials, truncated power functions and cone splines which can easily be defined by univariate *B*-splines (cf. the survey [10]).

TABLE I

Interpolation of f_1

n	d_n	ε_n	δ_n	$\tilde{\varepsilon}_n$	$\tilde{\delta}$
5	213	5.61×10^{-2}		4.98×10^{-2}	
6	291	1.23×10^{-2}		1.29×10^{-2}	
7	381	1.29×10^{-2}		1.40×10^{-2}	
8	483	5.22×10^{-3}		5.16×10^{-3}	
9	597	2.12×10^{-3}		3.57×10^{-3}	
10	723	1.57×10^{-3}	-5.2	2.03×10^{-3}	-4.6
11	861	1.04×10^{-3}		2.27×10^{-3}	
12	1011	9.57×10^{-4}	-3.7	1.01×10^{-3}	-3.7
13	1173	5.81×10^{-4}		7.75×10^{-4}	
14	1347	7.23×10^{-4}	-4.2	5.07×10^{-4}	-4.8
15	1533	5.09×10^{-4}		3.33×10^{-4}	
16	1731	4.02×10^{-4}	-3.7	2.41×10^{-4}	-4.4

We illustrate our methods by some numerical examples. We set $T=[0,1]\times[0,1]$ and consider the functions

$$f_1(x,y)=\tfrac{3}{4}e^{-((9x-2)^2+(9y-2)^2)/4}+\tfrac{3}{4}e^{-((9x+1)^2/49)-((9y+1)/10)}$$
$$+\tfrac{1}{2}e^{-((9x-7)^2+(9y-3)^2)/4}-\tfrac{1}{5}e^{-(9x-4)^2-(9y-7)^2}$$

and

$$f_2(x,y)=(y-x)_+^6,$$

where f_1 is the well-known Franke's testfunction and f_2 is a function in $C^5(T)\backslash C^6(T)$.

TABLE II

Interpolation of f_2

n	d_n	ε_n	δ_n	$\tilde{\varepsilon}_n$	$\tilde{\delta}$
5	213	3.35×10^{-5}		5.56×10^{-4}	
6	291	1.38×10^{-5}		2.97×10^{-4}	
7	381	6.41×10^{-6}		1.68×10^{-4}	
8	483	3.36×10^{-6}		1.01×10^{-4}	
9	597	1.90×10^{-6}		6.84×10^{-5}	
10	723	9.57×10^{-7}	-5.1	4.24×10^{-5}	-3.7
11	861	6.65×10^{-7}		2.97×10^{-5}	
12	1011	4.21×10^{-7}	-5.0	2.21×10^{-5}	-3.7
13	1173	2.64×10^{-7}		1.56×10^{-5}	
14	1347	1.97×10^{-7}	-5.0	1.07×10^{-5}	-4.0
15	1533	1.41×10^{-7}		8.72×10^{-6}	
16	1731	1.10×10^{-7}	-4.9	7.34×10^{-6}	-3.8

First, we interpolate the functions f_1 and f_2 by splines in $S_4^1(\Delta^1)$ using interpolation sets as in Fig. 2. Tables I and II show the cardinality d_n of the interpolation set, the corresponding error ε_n for $S_4^1(\Delta^1)$ and the decay exponent $\delta_n = \log(\varepsilon_n/\varepsilon_{n'})/\log(n/n')$, where $n' = n/2$ and $n = n_1 = n_2$. Moreover, as described in the section on data fitting, we interpolate the functions f_1 and f_2 on each subtriangle of the partition Δ^1 by a polynomial in $\tilde{\Pi}_3$ and then interpolate the resulting spline in $S_3^0(\Delta^1)$ by a spline in $S_4^1(\Delta^1)$. The following tables show the corresponding error $\tilde{\varepsilon}_n$, i.e. the deviation of the interpolating splines in $S_4^1(\Delta^1)$ from the functions f_1 and f_2, and the decay exponent $\tilde{\delta}_n = \log(\tilde{\varepsilon}_n/\tilde{\varepsilon}_{n'})/\log(n/n')$, where $n' = n/2$.

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